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Rapports de Recherche

N° 941

Programme 3

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Décembre 1988

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PERFORMANCE MODELLING OF QUEUES WITH RENDEZVOUS SERVICE

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October 27, 1988
Publication Interne n°435 - 24 Pages

Abstract: This paper describes a special queueing model for the performance of systems using the rendezvous mechanism. This mechanism is used in concurrent programming languages (Ada, Occam) and in some distributed operating systems (e.g. V). The distinctive aspect of rendezvous service is a second phase after the departure of the customer. We first study a single queue (which is a special case of a system studied by Skinner) with rendezvous, and its departure process. We term this an M/G+G/1 queue. Some small networks with rendezvous service are considered; they depart strongly from properties associated with product form queueing networks. A special case is described, however, in which the interdeparture times are "nearly" Poisson. A series of such queues has a behavior quite close to product form. This special case is a generalisation of the well-known M/M/1 queue to rendezvous service.

Keywords: Performance, rendezvous, distributed systems, queues, product form, Poisson process.

*The major part of this work was done while this author was visiting Carleton University, Ottawa, Canada.

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MODELISATION PAR FILES D'ATTENTE D'UN MECANISME DE RENDEZ-VOUS

Résumé:Nous proposons un modèle de file d'attente pour l'étude des performances de systèmes utilisant le mécanisme de rendez-vous. Cette notion est utilisée dans les langages parallèles Ada et Occam ainsi que le système d'exploitation V en informatique distribuée. Dans le cas d'un rendez-vous simple nous utilisons le modèle de Skinner (serveur avec vacance) où le service d'un client comprend 2 phases, la deuxième étant exécutée après le départ du client (notation:M/G+G/1). Avant l'étude de réseaux composés de telles files, nous étudions le processus de départ des clients et constatons qu'il tend vers un processus de Poisson si le service total est exponentiel et si l'utilisation du serveur tend vers 1. Une file possédant cette propriété (notée $M/M+\alpha M/1$) est ensuite analysée en isolation puis en réseau.Les résultats obtenus montrent que le comportement d'un tel réseau est très proche de celui d'un réseau à forme produit, cette propriété n'étant plus conservée pour des réseaux de files M/G+G/1.

Mots-clés:Performances, rendez-vous, systèmes distribués, files d'attente, forme produit, processus de Poisson.

1 Introduction

Rendezvous service occurs in distributed computer systems, when there is a handshake between a program sending a message and the program receiving it. This may be a feature of any system with cooperating tasks; it is enforced as a language feature of Ada ([1], [2]) and Occam [3], and it is an operating system feature in some distributed operating systems such as V [4]. Rendezvous may also occur between devices, as part of a hardware protocol. This paper derives performance results for systems containing rendezvous servers. The performance model for such a system is different from a standard queueing model, because of the rendezvous (abbreviated RNV). There is partial parallelism in execution between the customer (or caller) and the RNV server, for the server first executes while the caller waits, and then the caller leaves and the server continues for a time (see Figure 1). The first phase of service is sometimes termed synchronous or "in-rendezvous" service; the second phase is asynchronous or "post-rendezvous" service. This service pattern is shown in Section 2 to be a special case of the "walking type server" of Skinner [5], in which the second phase is a kind of vacation taken by the server after each service. Note that the notion of service phase in RNV queues is quite different from the phase-type service studied for instance by Neuts [6].

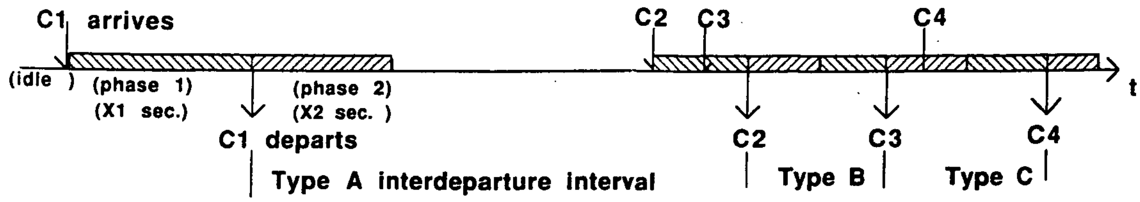

Figure 1: Two-Phase Service Pattern at a Rendezvous Server.

In computer systems we must be concerned with the behaviour of this queue within a system of queues. Other servers may be conventional queueing servers (processors, devices, asynchronous tasks), or may themselves be RNV servers. A different kind of network arises when a server which is a software task makes requests, while it is serving a customer. There can be deep chains of such servers; patterns like this have been termed "stochastic rendezvous nets" in [7], [8], [9], and several examples have been studied. This research has a more limited scope, and seeks to lay a foundation for understanding the single RNV server and its relationship to a few other servers. Figure 2 shows a RNV server and two small networks of RNV servers.

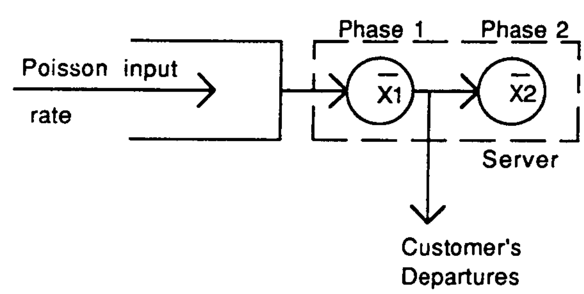

Figure 2: A single M/G+G/1 queue.

A single RNV queue, which might be associated with a server in a computer network, will be termed an M/G+G/1 queue. The M designates Poisson arrivals and the two G's designate two random service phases with general distributions. Section 2 analyzes the single M/G+G/1 queue. Section 3 analyzes its departure process, and shows that in certain cases, under heavy load, it may approach a Poisson process. Section 4 defines a special case, which we term M/M+ αM/1, with this property, and solves it completely. Section 5 considers this queue in a performance model with other servers, and evaluates approximate calculations based on product-form results for cases with a finite source, and queues in series.

2 The Single Rendezvous (M/G+G/1) Queue

The queue described in this section (see figure 4) is a special case of the queue with server "walking-time" studied by Skinner [5], Lavenberg [10], Gelenbe and Mitrani [11] and (with general arrivals) by Gelenbe and Iasnogorodski [12]. Figure 3 shows the sequence of operations by this server.

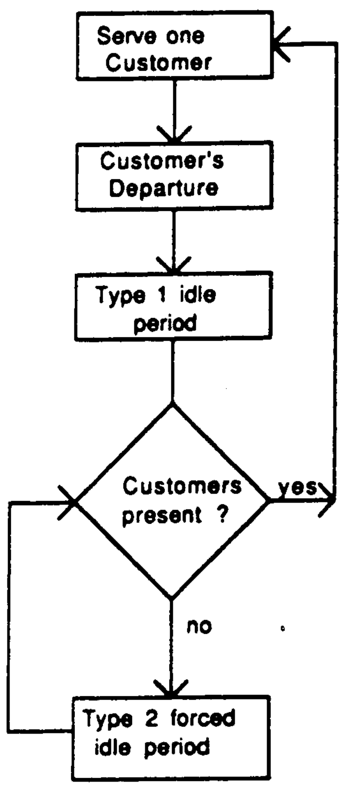

Fig 3:Skinner's "Server of Walking Type"

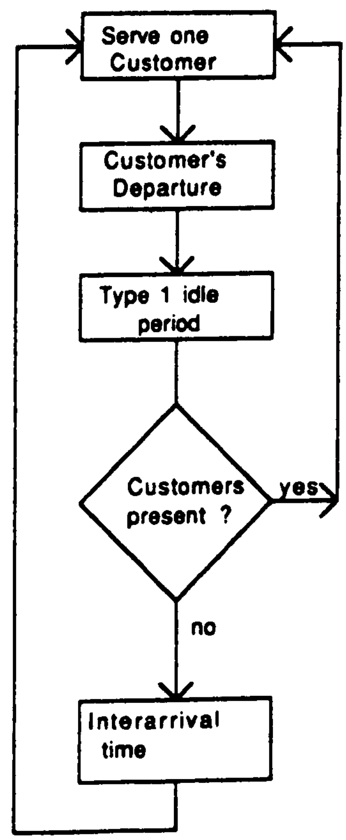

Fig 4:Adaptation to Rendezvous service

Suppose that the arrival is Poisson with rate λ. Each time a service ends and the queue is not empty, the server serves one customer for a service time x_1 and then takes a rest period x_2, after which it returns to examine the queue again. If it discovers that the queue is empty, the server goes away for a random absence period y, after which it will return once again to examine the queue. The mean waiting time, obtained by Skinner, is

$$\overline{w} = \frac{\overline{y^2}}{2\overline{y}} + \frac{\lambda \overline{x^2}}{2(1-\lambda\overline{x})} + \overline{x_1} \tag{1}$$

where $x = x_1 + x_2$, and the bar denotes expectation. Skinner also derived the delay distribution and the probability distribution for the queue size, as transforms.

Our RNV server is similar, but without the absence y; Figure 4 shows the operations. To adapt the result (1) we notice that the first term is the mean residual life of the absence y; if $y = 0$ then this term is zero, giving

$$\overline{w} = \frac{\lambda \overline{x^2}}{2(1-\lambda\overline{x})} + \overline{x_1} \tag{2}$$

One can easily see that if $x_2 = 0$ and $x_1 = x$, this last expression is the Pollaczek-Khintchine formula for the M/G/1 queue.

If the two phases are exponential (M/M+M/1 queue), Woodside and Neron [13] derived more detailed results, including the probability for the phase of the server. Using the following notation:

$p(n,i)$ = steady state probability that the queue contains n customers and the service phase is i;

$p(n) = p(n,1) + p(n,2)$ = the probability of n customers;

$\mu_i = 1/\overline{x_i}$, $i = 1,2$ for the phase service rates;

$\rho = \lambda(\overline{x_1} + \overline{x_2})$;

$Q_i(z) = \sum_{n=0}^{\infty} p(n,i) z^n$ = Z-transform of the sequence $p(n,i)$, $n \geq 0$,

they obtained:

$$Q_1(z) = \frac{\mu_1(1-\rho)(\mu_2 - \lambda z)}{(\mu_1 + \lambda(1-z))(\mu_2 - \lambda z) - \mu_2 \lambda} \tag{3}$$

$$Q_2(z) = \frac{\lambda}{\mu_2 - \lambda z} Q_1(z) \tag{4}$$

3 The Departure Process

In a distributed computer system it may be necessary to analyse networks of service centers, including some with RNV service. Then the departure process from a RNV server will influence the behavior of other servers in the network. In particular if the departure process were Poisson then the queue would have the "Poisson ⇒ Poisson" property, and [14] shows that the RNV servers could be included in product form queueing networks. This section shows that in a very special case, the departure process may be nearly Poisson.

Define:

$b_i(x) =$ the service time x_i in phase i, $i = 1, 2$;

$B_i(s) =$ the Laplace transform of $b_i(x)$;

$D(s) =$ the Laplace transform of $d(t)$, the interdeparture time;

$p_A(0,1) =$ the probability that the server is idle at an arrival instant;

$p_A(0,2) =$ the probability that the server is executing its second phase, with no job waiting, at an arrival;

$p_D(0) =$ the probability that a departure leaves no other customer behind.

We begin by determining the distribution of interdeparture times. An interdeparture interval is one of three types, as shown in Figure 1. If the arrival that ends it comes to an idle server, we will call it a Type A interval and it is made up of a phase 2 service, an interarrival interval, and a phase 1 service, in that order. Otherwise, if there is always a customer in the system, the server is continuously busy and the interval is made up of a phase 2 service and a phase 1 service (call this a Type B interval). Finally, an interval may begin with no customer waiting, but one arrives before phase 2 is over. This is a Type C interval.

The probabilities of the three types of interval are found by considering what happens following a departure. With probability $p_D(0)$ there is no job left behind when the second phase begins. The probability of no arrival while the second phase is executed is easily found to be $B_2(\lambda)$; thus the probability of a Type A interval is $p_D(0)B_2(\lambda)$. For Type B it is unity minus this amount. Also the probability that an arrival ends a Type A interval (and finds the server idle) is $p_A(0,1)$, and must be the same as the probability that a departure begins one, so that:

$$p_A(0,1) = p_D(0)B_2(\lambda)$$

$$p_D(0) = p_A(0,1)/B_2(\lambda) \tag{5}$$

Because the arrivals are Poisson, the probabilities seen by an arrival are the steady-state probabilities; thus

$$p_A(0,1) = p(0,1) \tag{6}$$

Also, the server is idle with probability:

$$p(0,1) = 1-\rho$$

If we combine this with (5), we find

$$p_D(0) = p(0,1)/B_2(\lambda) = (1-\rho)/B_2(\lambda) \tag{7}$$

$$\text{Prob}(\textit{Type A interval}) = p_D(0)B_2(\lambda) = (1-\rho)$$

$$\text{Prob}(\textit{Type B interval}) = 1 - p_D(0) = 1 - (1-\rho)/B_2(\lambda)$$

$$\text{Prob}(\textit{Type C interval}) = p_D(0)(1-B_2(\lambda)) = (1-\rho)(1-B_2(\lambda))/B_2(\lambda)$$

The density function of each type of inter-departure interval will now be determined. In a type A interval the second phase of service has density $d_A(x_2)$, which is $b_2(x_2)$ conditioned on the fact that x_2 is less than the time to the next arrival. Thus:

$$d_A(x_2) = b_2(x_2)e^{-\lambda x_2}/B_2(\lambda)$$

which has the transform $B_2(s+\lambda)/B_2(\lambda)$. The Type A interval is the sum of this conditional phase-2 service, an interarrival period, and a phase-1 service. Its density has the transform $D_A(s)$:

$$D_A(s) = A(s)B_1(s)B_2(s+\lambda)/B_2(\lambda)$$

The Type B interval is simply the sum of two phases, with transform $D_B(s) = B_1(s)B_2(s)$. The Type C interval includes a phase-2 service conditioned on an arrival coming before it ends; its conditional density is $d_C(x_2)$ given by

$$d_C(x_2) = b_2(x_2)(1 - e^{-\lambda x_2})/(1 - B_2(\lambda))$$

with transform $(B_2(s) - B_2(s+\lambda))/(1 - B_2(\lambda))$. Thus the Type C interval consists of a conditional second phase, plus a first phase. Its density has the transform $D_C(s)$:

$$D_C(s) = B_1(s)[B_2(s) - B_2(s+\lambda)]/(1 - B_2(\lambda)$$

The overall interdeparture distribution is governed by

$$D(s) = (1-\rho)D_A(s) + [1 - (1-\rho)/B_2(\lambda)]D_B(s) + (1-\rho)[1 - B_2(\lambda)]D_C(s)/B_2(\lambda)$$

$$D(s) = (1-\rho)A(s)B_1(s)\frac{B_2(s+\lambda)}{B_2(\lambda)} + [1 - \frac{(1-\rho)}{B_2(\lambda)}B_1(s)B_2(s)$$
$$+(1-\rho)B_1(s)\frac{[B_2(s) - B_2(s+\lambda)]}{B_2(\lambda)}$$

With a little algebra, using $A(s) = \lambda/(s+\lambda)$, this reduces to

$$D(s) = B_1(s)B_2(s) - \frac{(1-\rho)s}{B_2(\lambda)(s+\lambda)}B_1(s)B_2(s+\lambda) \tag{8}$$

From (8), if the first phase is exponential with $B_1(s) = \mu/(s+\mu)$ and the second phase vanishes with $B_2(s) = 1$, we can verify that $D(s) = \lambda/(s+\lambda)$ as it should for an M/M/1 queue.

Also from (8), it is clear that, as $\rho \to 1$, $D(s) \to B_1(s)B_2(s)$; and the interdeparture interval is simply the sum of two phase times. Now consider a special case where $\rho \to 1$ and the sum of the two phases together has an exponential distribution,

$$B_1(s)B_2(s) = \mu/(s+\mu) \tag{9}$$

with $\mu = 1/(\overline{x_1} + \overline{x_2})$. Because the independence of the phase service times, the departure process approaches a process with independent and exponentially distributed intervals – i.e. a Poisson process.

4 The M/M + αM/1 Queue

The following service process has been invented to make the sum of the phases exponential in order to study these cases further. It has exponential first and second phases, and a probability $1-\alpha$ of skipping the second phase alltogether (see Figure 5).

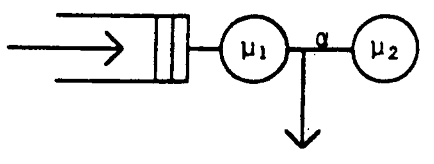

Figure 5:The M/M+αM/1 queue.

To make the sum of the two phases exponential, their rates μ_1 and μ_2 are forced to satisfy:

$$\mu_2 = \mu_1(1-\alpha) \tag{10}$$

Then $B_1(s) = \mu_1/(\mu_1+s)$, $B_2(s) = \mu_2/(\mu_2+s)$, and if we note B(s) the Laplace transform of the total service time, we obtain:

$B(s) = (1-\alpha)B_1(s) + \alpha B_2(s)$, hence:

$$B(s) = \frac{\mu_1((1-\alpha)(\mu_2+s)+\alpha\mu_2)}{(\mu_1+s)(\mu_2+s)} \tag{11}$$

After some reduction this gives:

$$B(s) = \mu_2/(\mu_2+s)$$

which shows that the sum is exponentially distributed, with mean $\mu^{-1} = \mu_2^{-1}$.

We shall term this special case an M/M + αM/1 queue. We next derive the steady-state probabilities for this queue.

The steady-state probabilities

The notation e= (n,i) will be used for the state with the server in phase i and n customers in the queue. The global balance equations, or Chapman-Kolmogoroff equations, which say that the rate of flow out of a state e is equal to the rate of flow into the same state, give (see figure 6)

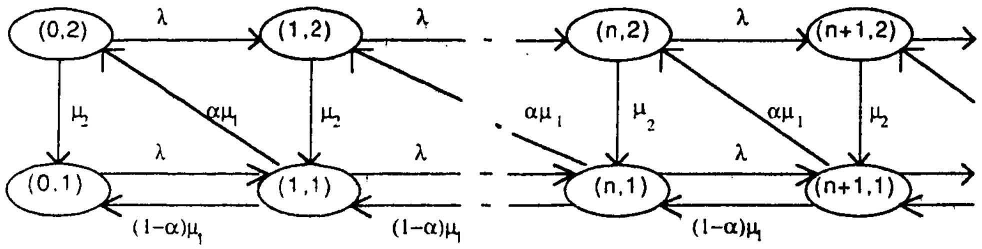

Figure 6: State diagramm of a M/M+α M/1 queue

e=$(0,1)$: $p(0,2)\mu_2 + p(1,1)\mu_1(1-\alpha) = p(0,1)\lambda$

$$\Leftrightarrow \; p(0,2)\mu_2 + p(1,1)\mu_2 = p(0,1)\lambda \quad (E_0)$$

e=(0,2): $p(1,1)\mu_1\alpha = p(0,2)(\mu_2 + \lambda)$

$$\Leftrightarrow \; p(1,1)(\mu_1 - \mu_2) = p(0,2)(\mu_2 + \lambda) \quad (E_0')$$

e=$(n,1)$: $p(n-1,1)\lambda + p(n,2)\mu_2 + p(n+1,1)\mu_1(1-\alpha) = p(n,1)(\mu_1 + \lambda)$

$$\Leftrightarrow \; p(n-1,1)\lambda + p(n,2)\mu_2 + p(n+1,1)\mu_2 = p(n,1)(\mu_1 + \lambda) \quad (E_n)$$

e=$(n,2)$: $p(n-1,2)\lambda + p(n+1,1)\mu_1\alpha = p(n,2)(\mu_2 + \lambda)$

$$\Leftrightarrow \; p(n-1,2)\lambda + p(n+1,1)(\mu_1 - \mu_2) = p(n,2)(\mu_2 + \lambda) \quad (E_n')$$

Using the z-transforms $Q_1(z)$ of $(p(n,1))$ $n \geq 0$ and $Q_2(z)$ of $(p(n,2))$ $n \geq 0$ we obtain, after multiplying the equations (E_n) $n \geq 0$ by z^n:

$$Q_1(z)[\lambda z + \frac{\mu_2}{z} - \mu_1 - \lambda] + \mu_2 Q_2(z) = (\frac{\mu_2}{z} - \mu_1)p(0,1)$$

$$Q_1(z)[\lambda z^2 - \mu_1 z + \mu_2 - \lambda z] + \mu_2 z Q_2(z) = (\mu_2 - \mu_1 z)p(0,1)$$

Using the same method with the equations (E_n') $n \geq 0$, we obtain:

$$(\mu_1-\mu_2)Q_1(z)+Q_2(z)[\lambda z^2-(\mu_2+\lambda)z]=(\mu_1-\mu_2)p(0,1)$$

To obtain $Q_1(z)$ and $Q_2(z)$, we have to solve the system:

$$\begin{cases} Q_1(z)(\lambda z^2-z(\mu_1+\lambda)+\mu_2)+\mu_2 z Q_2(z)=(\mu_2-\mu_1 z)p(0,1) \\ Q_1(z)(\mu_1-\mu_2)+Q_2(z)(\lambda z^2-(\mu_2+\lambda)z)=(\mu_1-\mu_2)p(0,1) \end{cases}$$

After some algebra, this gives:

$$Q_1(z)=\frac{(\mu_2(\lambda+\mu_1)-\lambda\mu_1 z)p(0,1)}{(\lambda z-\mu_2)(\lambda z-(\lambda+\mu_1))}$$

$$=\frac{\lambda\mu_2 p(0,1)}{(\lambda+\mu_1-\mu_2)(\mu_2-\lambda z)}+\frac{(\lambda+\mu_1)(\mu_2-\mu_1)p(0,1)}{(\lambda+\mu_1-\mu_2)(\lambda z-(\lambda+\mu_1))} \tag{12}$$

$$Q_2(z)=\frac{\lambda(\mu_1-\mu_2)p(0,1)}{(\lambda z-\mu_2)(\lambda z-(\lambda+\mu_1))}$$

$$=\frac{\lambda(\mu_1-\mu_2)p(0,1)}{(\lambda z-\mu_2)(\mu_2-\mu_1-\lambda)}+\frac{\lambda(\mu_1-\mu_2)p(0,1)}{(\lambda+\mu_1-\mu_2)(\lambda z-(\lambda+\mu_1))} \tag{13}$$

After that, one can easily expand $Q_1(z)$ and $Q_2(z)$ in a power-series where the coefficients of z^n are the terms $(p(n,1))$ $n\geq 1$ and $(p(n,2))$ $n\geq 0$:

$$p(n,1)=[\frac{\lambda}{\lambda+\mu_1-\mu_2}(\frac{\lambda}{\mu_2})^n+\frac{\mu_1-\mu_2}{\lambda+\mu_1-\mu_2}(\frac{\lambda}{\lambda+\mu_1})^n]p(0,1) \tag{14}$$

$$p(n,2)=[\frac{(\mu_1-\mu_2)}{(\lambda+\mu_1-\mu_2)}(\frac{\lambda}{\mu_2})^{n+1}+\frac{(\mu_2-\mu_1)}{(\lambda+\mu_1-\mu_2)}(\frac{\lambda}{\lambda+\mu_1})^{n+1}]p(0,1) \tag{15}$$

This solution has a close relationship to the M/M/1 case. If we form the sum $\sum_{n=0}^{\infty}[p(n,1)+p(n,2)]=1$, we find, after some algebra, that

$$p(n,1)+p(n-1,2)=(\lambda/\mu_2)^n p(0,1)....n=0,1,...$$

$$p(0,1)=1-\lambda/\mu_2=1-\rho$$

which is similar to the well-known result $p(n)=(\lambda/\mu_2)^n p(0)$ for the M/M/1 queue with arrival rate λ and service rate μ_2. But in our model the state $(n-1,2)$ does not correspond to n customers. It is as if a fictitious customer is created in the beginning of the second phase and deleted at the end of this phase.

Ergodicity

If $(\lambda/\mu_2) < 1$ and $\lambda/(\lambda+\mu_1) < 1$ i.e. $\lambda < \mu_2$, then the sum $\sum_{n=0}^{\infty}(p(n,1)+p(n,2))$ is finite and the steady-state probabilities exist. To prove that these conditions are necessary, one can see that μ_1 is always strictly positive; so to have $\sum_{n=0}^{\infty} p(n,1)$ finite, the series $\sum(\lambda/\mu_2)^n$ must converge, for which $\lambda < \mu_2$. A necessary and sufficient condition for the ergodicity of the queue M/M+αM/1 is then $\lambda < \mu_2$.

Limiting Cases M/M/1 and M/0+M/1

If $\alpha = 0$, there is no second phase, and we obtain exactly the M/M/1 queue with service rate $\mu_1 = \mu_2 = \mu$. From (14) and (15), setting $\alpha = 0$ and $\mu_1 = \mu_2 = \mu$, we obtain the standard results in the form $p(n,1) = (1-\lambda/\mu)(\lambda/\mu)^n$, and $p(n,2) = 0$. Thus the M/M+ αM/1 queue is a generalization of the M/M/1 case with RNVs and with the degree of departure from M/M/1 controlled by the parameter α.

On the other hand, if $\mu_1 \to \infty$ then $\alpha \to 1$ and there is no first phase (its service rate $\to \infty$). We will term this case M/0+M/1. The first-phase service time of a customer is zero and he leaves the queue at the beginning of his service. There is always a second phase of rate μ_2. The solution is:

$$p(n,1) = 0\,; \qquad n \geq 1$$
$$p(n,2) = (\frac{\lambda}{\mu_2})^{n+1}(1-\frac{\lambda}{\mu_2})\,; \quad n \geq 0$$

This corresponds to an M/M/1 queue in which the service is performed in the absence of the customer. This model represents the behaviour of a process written in the Occam language [3] to run on the transputer. In Occam a process sending a message over a channel to another transputer waits until the message is received, and then resumes in parallel with the receiver. The processing on the message is done after the sender leaves the queue, just as in the M/0+M/1 queue.

Probability of finding n customers in the queue

According to the relationships (14) and (15) we find

$$
\begin{aligned}
p(n) &= p(n,1) + p(n,2) \\
&= \left(\frac{\mu_1}{\lambda+\mu_1-\mu_2}\left(\frac{\lambda}{\mu_2}\right)^{n+1} + \frac{\mu_1(\mu_1-\mu_2)}{\lambda(\lambda+\mu_1-\mu_2)}\left(\frac{\lambda}{\lambda+\mu_1}\right)^{n+1}\right)p(0,1) \\
&= \frac{\mu_1}{\lambda}\left[\frac{\lambda}{\lambda+\mu_1-\mu_2}\left(\frac{\lambda}{\mu_2}\right)^{n+1} + \frac{\mu_1-\mu_2}{\lambda+\mu_1-\mu_2}\left(\frac{\lambda}{\lambda+\mu_1}\right)^{n+1}\right]\left(1-\frac{\lambda}{\mu_2}\right) \qquad (16)
\end{aligned}
$$

for $n \geq 0$.

Remember that $p(0)$ is $p(0,1) + p(0,2)$ and is not equal to $1 - (\lambda/\mu_2)$. One can easily note that $p(n) = (\mu_1/\lambda)p(n+1,1)$.

5 Finite Source M/M+αM/1 queue

A RNV server in a computer network is better represented with a finite population of users, as indicated in Figure 7.

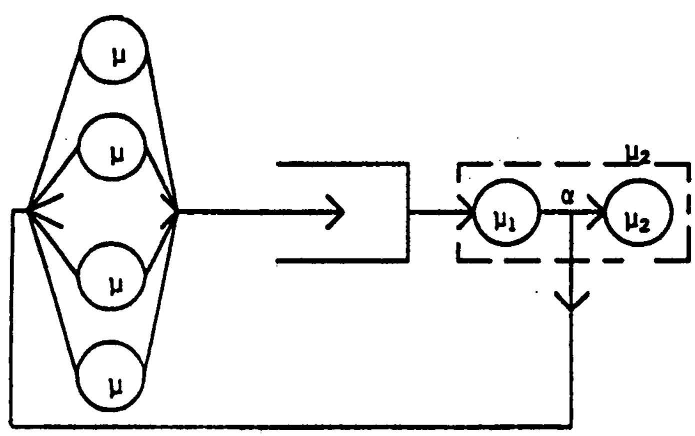

Figure 7

This is also the simplest form of network involving a RNV server. This section gives an exact Markov chain model and compares it to an approximation based on the notion of product form.

The network has N customers and two stations S and S'. S is an "infinite server" station with service rate μ and S' is an M/M + αM/1 RNV server as described in section 4.

A state of this system e has the following form:

$\mathbf{e} = (n, n', i)$ with

n: number of customers in S

n': number of customers in S' $(n' = N - n)$

i: phase 1 or 2.

For the computation of the steady-state probabilities we first establish the global balance equations:

$\mathbf{e}$ = (N,0,1):

$$p(N-1,1,1)\mu_1(1-\alpha) + p(N,0,2)\mu_2 = p(N,0,1)N\mu \quad ...(F_0)$$

$\mathbf{e}$ = (N,0,2):

$$p(N-1,0,2)\mu_1\alpha = p(N,0,2)(\mu_2 + N\mu) \quad (F_0')$$

$\mathbf{e}$ = (N-n,n,1):

$$p(N-n-1,n+1,1)\mu_1 + p(N-n+1,n-1,1)(N-n+1)\mu + p(N-n,n,2)\mu_2$$
$$= p(N-n,n,1)(\mu_1 + (N-n)\mu) \quad ...(F_n)$$

$\mathbf{e}$ = (N-n,n,2):

$$p(N-n-1,n+1,1)\mu_1\alpha \quad + p(N-n+1,n-1,2)(N-n+1)\mu$$
$$= p(N-n,n,2)((N-n)\mu + \mu_2) \quad ...(F_n')$$

$\mathbf{e}$ = (0,N,1):

$$p(1,N-1,1)\mu + p(0,N,2)\mu_2 = p(0,N,1)\mu_1 \quad ...(F_N)$$

$\mathbf{e}$ = (0,N,2):

$$p(1,N-1,2)\mu = p(0,N,2)\mu_2 \quad ...(F_N')$$

To proceed we will use the simpler notation;

$$u(n) = p(N-n, n, 1)$$

$$v(n) = p(N-n, n, 2) \text{ ; for each } n \geq 0$$

With this, and replacing $\mu_2 = \mu_1(1-\alpha)$ we get the system:

$$u(0)N\mu - v(0)\mu_1(1-\alpha) + u(1)\mu_1(1-\alpha) = 0 \quad .....(F_0)$$

$$u(1)\mu_1\alpha - v(0)(\mu_1(1-\alpha) + N\mu) = 0 \quad ....(F_0')$$

$$u(n-1)(N-n+1)\mu - u(n)(\mu_1 + (N-n)\mu) + v(n)\mu_1(1-\alpha)$$

$$+u(n+1)\mu_1(1-\alpha) = 0 \quad ....(F_n)$$

$$v(n-1)(N-n+1)\mu - v(n)((N-n)\mu + \mu_1(1-\alpha)) + u(n+1)\mu_1\alpha = 0 \; ...(F_n')$$

$$u(N-1)\mu - u(N)\mu_1 + v(n)\mu_1(1-\alpha) = 0 \quad ...(F_N')$$

$$v(N-1)\mu - v(N)\mu_1(1-\alpha) = 0 \quad (F_N')$$

and

$$\sum_{n=0}^{N}(u(n) + v(n)) = 1 \quad ...(G)$$

Replace (F_n') by $(F_n'') = (F_n) - (1-\alpha)/(\alpha)(F_n')$, for $n = 1, ...N-1$, to give:

$$u(n-1)(N-n+1)\mu - v(n-1)(N-n+1)\mu(1-\alpha)/\alpha$$

$$+u(n)(\mu_1 + (N-n)\mu) - v(n)((N-n)\mu + \mu_1(1-\alpha))/\alpha = 0 \quad ...(F_n'')$$

Due to the fact that the network is markovian and irreducible, the solution of this system is unique but a closed form is not easy to obtain because of the state dependency of the coefficients.

However, we can easily obtain the solution numerically as follows:

Step 1: Set $u(0) = 1$, and find $v(0)$ and $u(1)$ from equations (F_0) and (F_0').

Step 2: For $n = 1$ to $N-1$, calculate $v(n)$ from equation (F_n''); $u(n+1)$ from (F_{n+1}).

Step 3: $v(N)$ is deduced from equation (F_N)

Step 4: Find $u(0)$ using equation (G).

Numerical results were obtained for 12 customers, $\mu_1 = 1$, and various values of μ and α. The throughputs λ found numerically are given in Table 1. With this server and sufficiently heavy traffic a product form approximation should give reasonable accuracy. A suitable product form approximation here would be a classical M/M/1/N queue, for which the state probabilities are well-known to be:

$$p(n) = \frac{c}{(N-n)!}\left(\frac{\mu}{\mu_2}\right)^n \quad n = 0, 1, 2, \ldots$$

where n is a reduced form for the state $(N-n, n)$ and c is a normalizing constant.

Table 1 also shows throughputs for the product-form approximation; the agreement is extremely good, except for very small μ and intermediate values of α. Figure 8 shows the comparison for $\mu = 0.01$, against α. These were the only results in the Table with perceptible differences between RNV service and the analytic approximation. We notice how, as $\alpha \to 0$ the agreement becomes good (as the server becomes a simple exponential server). In general the agreement is good if α is small or if $\mu_2/(12\mu)$ is large (a condition derived from bound analysis, indicating cases where the server is relatively heavily utilized). The response time can be found from the Table using Little's formula, as $w = (12/\lambda) - \mu^{-1}$.

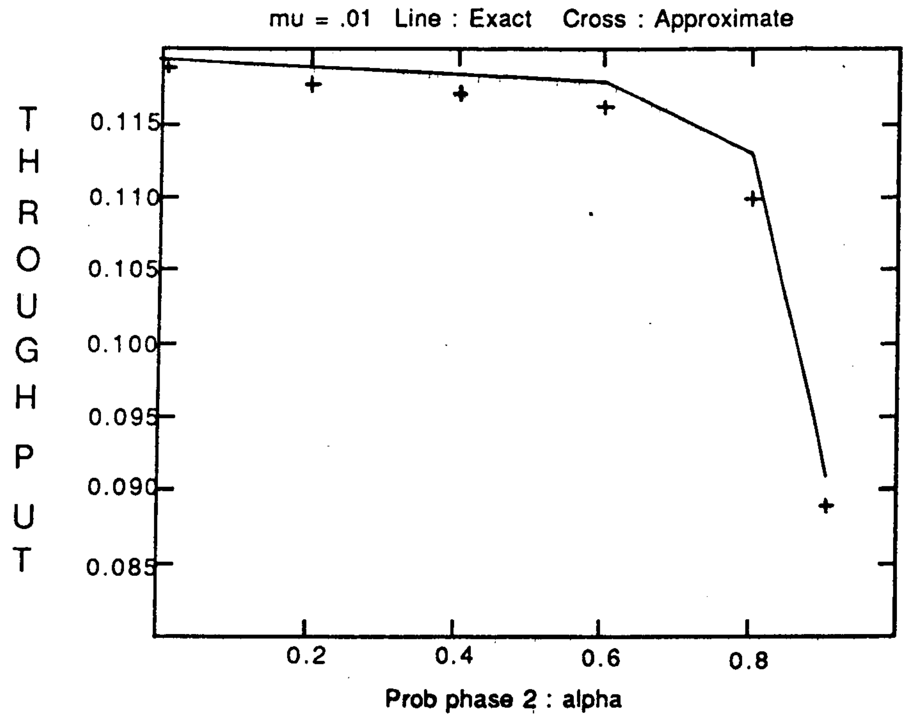

Figure 8: Finite-Source Example; Results for μ=0.01

6 RNV Servers in Series

A series of RNV queues fed by a Poisson source of jobs will not obey product form because the outputs are not Poisson. This section examines the amount of the difference for several cases, by comparing simulation results for the delays through three queues in series, to an analytic calculation based on the formula of Skinner. The system is shown in Figure 9.

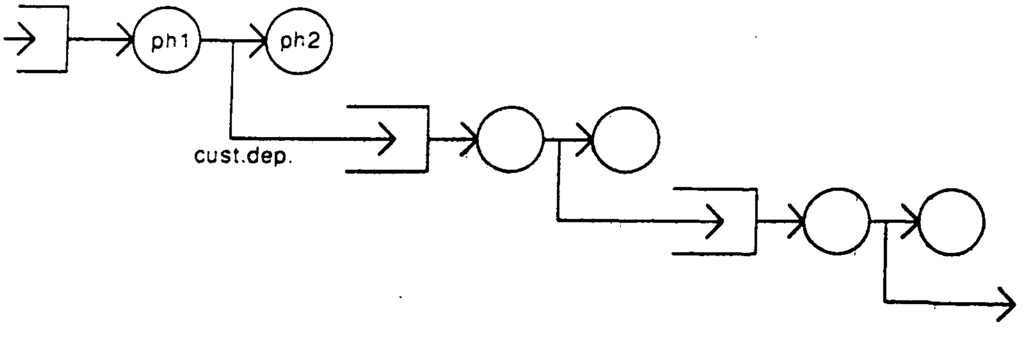

Figure 9.

Table 1: Throughputs for Finite-Source M/M+αM/1/N Server

μ / α		0.01	0.05	0.1	0.3	0.5	0.7	0.9
0.01	exact	0.1190	0.547	0.875	0.99	0.59	0.99	0.99
	approx.	0.1187	0.547	0.875	0.99	0.53	0.99	0.99
0.2	exact	0.1186	0.530	0.762	0.800	0.800	0.800	0.800
	approx.	0.1183	0.526	.0759	0.800	0.800	0.800	0.800
0.4	exact	0.1186	0.488	0.594	0.600	0.600	0.600	0.600
	approx.	0.1176	0.481	0.593	0.600	0.600	0.600	0.600
0.6	exact	0.1177	0.384	0.399	0.399	0.400	0.400	0.400
	approx.	0.1161	0.379	0.399	0.399	0.400	0.400	0.400
0.8	exact	0.1131	0.200	0.200	0.200	0.200	0.200	0.200
	approx.	0.1096	0.200	0.200	0.200	0.200	0.200	0.200
0.9	exact	0.0906	0.100	0.100	0.100	0.100	0.100	0.100
	approx.	0.0880	0.100	0.100	0.100	0.100	0.100	0.100
0.99	exact	0.010	0.010	0.010	0.010	0.010	0.010	0.010
	approx.	0.010	0.010	0.010	0.010	0.010	0.010	0.010

Table 2: Delays in Rendezvous Queues in Series

(cases (a),(b),(c) have $\mu_2 = 1$, $\overline{x_1} = 1 - \alpha$)

		Simulation Delay (±95% conf.int)	Analytic Delay Calculation	% error
Case(a)	queue 1	2.301(±0.030)	2.300	0.04
/M+αM/	queue 2	2.369(±0.031)	2.300	1.4
$\alpha = 0.2$	queue 3	2.254(±0.024)	2.300	2.0
	Total			
Case(b)	queue 1	2.083(±0.087)	2.10	0.82
/M+αM/	queue 2	2.059(±0.077)	2.10	2.0
$\alpha = 0.4$	queue 3	1.918(±0.095)	2.10	0.5
	Total	6.061(±0.221)	6.30	3.9
Case(c)	queue 1	1.900(±0.080)	1.9	0
/M+αM/	queue 2	1.854(±0.086)	1.9	2.5
$\alpha = 0.6$	queue 3	1.805(±0.053)	1.9	5.3
	Total	5.559(±0.176)	5.7	2.5
Case(d)	queue 1	1.622(±0.015)	1.625	0.18
/M+M/	queue 2	1.370(±0.017)	1.625	19
$\overline{x_1} = \overline{x_2}$=0.5	queue 3	1.304(±0.017)	1.625	25
	Total	4.296(±0.040)	4.875	23
Case(e)	queue 1	1.368(±0.031)	1.37	0.14
Uniform(0,1)	queue 2	0.926(±0.025	1.37	47
$\overline{x_1} = \overline{x_2}$=0.5	queue 3	0.825(±0.031)	1.37	66
	Total	3.119(±0.070)	4.11	32

The analytic calcultaion for the delay w at each queue used the average input rate λ, and the formula (1) for the delay. Five cases were simulated, all with $\lambda = 0.6$ and the total server time $\overline{x} = \overline{x_1} + \overline{x_2} = 1$ at each queue. The cases are:

(a) M/M+αM/1, with $\alpha = 0.2$; (b) M/M+αM/1, with $\alpha = 0.4$;

(c) M/M+αM/1, with $\alpha = 0.6$; (d) M/M+M/1, with $\overline{x_1} + \overline{x_2} = 0.5$;

(e) M/G+G/1, with both phase service times uniform on the interval $(0, 1)$.

In all cases in Table 2 the delay for queue 1 (which has Poisson input and matches the assumptions of (1) exactly) is well within 95% confidence intervals. In cases (a), (b), and (c) queue 2, with nearly Poisson arrivals, still agrees quite well although the input is not Poisson; queue 3 diverges more. The disagreement at queues 2 and 3 is much greater in cases (c) and (d). This shows that the M/M+αM/1 RNV queue is much more susceptible to approximate analysis based on product form, than other RNV queues in series. We hypothesize that the same is true of other RNV queues satisfying equation (9).

7 Conclusions

Basic properties of a single M/G+G/1 rendezvous (RNV) queue have been assembled, being a combination of new results and applications of known results. From considerations of the departure process a special class of RNV queues was found with nearly Poisson departures. One queue in this class, a RNV generalisation of the M/M/1 queue, was defined and termed the M/M+αM/1 queue; a solution for its state probabilities is included, and a discussion of its properties.

An approximation for queueing networks containing ./M+αM/1 service centers was tested on simple cases and appears to be promising. The approximation is a product-form expression using Skinner's solution for product term for the ./M+αM/1 centers, and is motivated by the fact that in heavy traffic the centers approximately satisfy the "Poisson in $\Rightarrow$Poisson out"criterion of [14] for product form. For a finite source system and for servers in series with utilizations as low as 0.6, good accuracy was obtained.

Thus the ./M+αM/1 RNV service pattern is an important special case in providing solvable system.

Acknowledgements

Discussions with J-Y Le Boudec, B. Sericola, K. Si Smail, E. Neron, and M. Neuts helped to clarify some of the issues in the analysis. The financial support of the Natural Sciences and Engineering Research Council of Canada, under its operating grants program, and the Ministry of External Relations of France is gratefully acknowledged.

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